LAX EQUATIONS, SINGULARITIES AND RIEMANN-HILBERT PROBLEMS

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ABSTRACT. The existence of singularities of the solution for a class of Lax equations is investigated using a development of the factorization method first proposed by Semenov-Tian-Shansky and Reymann [11], [9]. It is shown that the existence of a singularity at a point $t = t_i$ is directly related to the property that the kernel of a certain Toeplitz operator (whose symbol depends on t) be non-trivial. The investigation of this question involves the factorization on a Riemann surface of a scalar function closely related to the above-mentioned operator. An example is presented and the set of singularities is shown to coincide with the set obtained by classical methods. This comparison involves relating the two Riemann surfaces associated to the system by these methods.

1. INTRODUCTION

In this paper we investigate the existence of singularities of the solutions of Lax equations for a class of equations that applies to most finite-dimensional dynamical systems such as *e.g.* classical tops (see *e.g.* [3], [9], [10]). To that end we consider the time variable t to be a complex variable and determine the singularities of the solution in the complex plane. This is tied to the question of global existence of solutions for real t as the non-existence of singularities for real t implies global existence of the solution. Also, it is likely that full knowledge of the location of complex singularities may eventually give more insight into the dynamics of the system.

Our approach is a development of the factorization method first proposed by Semenov-Tian-Shansky and Reymann [11], [9] which in turn may be seen as a generalization of the AKS (Adler-Kostant-Symes) theorem that applies to finite dimensional algebras [1].

To the best of our knowledge the first application of this method in the setting of an infinite-dimensional algebra appeared in [4], which focused on a restricted class of Lax equations. The absence in the literature of more fully computed examples of application of this method is probably due to the fact that it involves the factorization of a continuous function on a contour in a Riemann surface (for a general treatment of this problem see [5]).

Considering t as a complex variable and extending the class of Lax equations requires a new analysis of the results of [4], where we were able to avoid making some delicate assumptions like the differentiability

of the factors in the Wiener-Hopft factorization (see definition below) of the matrix function $\exp(tL_0)$, where L_0 is the value of the Lax matrix L_t at $t = 0$. This is done in Section 2 and continued in Section 3 for the question of location of the singularities of the solution. Our approach makes the treatment fully rigorous and, in our view, is crucial for the treatment in the context of a complex t.

The main result is Theorem 3.1, which we state next. For this we note that the space $[C_\mu(S^1)]^n$ of Hölder continuous $n \times n$ matrix-valued functions has a direct sum decomposition

$$[C_\mu(S^1)]^n = \left[C_\mu^+\right]^n \oplus \left[C_\mu^-\right]_0^n, \tag{1.1}$$

where $\left[C_\mu^+\right]^n$ is the subspace [1] of functions having an analytic extension to the unit disc $\mathbb{D}$ and $\left[C_\mu^-\right]_0^n$ is the subspace of functions admitting analytic extensions to $\mathbb{C} \setminus \overline{\mathbb{D}}$ that vanish at infinity. In what follows $\left[C_\mu^-\right]^n = \left[C_\mu^-\right]_0^n \oplus \mathbb{C}^n$.

Consider the Toeplitz operator

$$T_G = P^+ G I_+ \colon \left[C_\mu^+\right]^n \to \left[C_\mu^+\right]^n \tag{1.2}$$

where $G = \exp(tL_0)$, I_+ is the identity operator on $\left[C_\mu^+\right]^n$ and P^+ is the projection of $[C_\mu]^n$ onto $\left[C_\mu^+\right]^n$ associated to the decomposition (1.2). Theorem 3.1 states that the Lax equation,

$$\frac{dL_t}{dt} = \left[L_t^+ + A_0, L_t\right], \tag{1.3}$$

has a solution in a neighbourhood of the point t_i *iff* T_G is injective at the point t_i. In the above equation $A_0 = P_0 L_t$ where $P_0 \colon [C_\mu(S^1)]^{n\times n} \to \mathbb{C}^{n\times n}$ is a bounded linear operator.

In the calculation of the singularities we need the notion of a Wiener-Hopf factorization. Let $G \colon S^1 \to [C_\mu]^{n\times n}$. We say that G possesses a *Wiener-Hopf factorization* (also called Riemann-Hilbert factorization and Birkhoff factorization [7], [8]) if G can be represented in the form

$$G = G_- D G_+, \tag{1.4}$$

where $G_\pm$ and their inverses belong to $\left[C_\mu^\pm\right]^{n\times n}$ and $D = \mathrm{diag}(r^{k_1}, \ldots, r^{k_n})$ with $k_1 \geq k_2 \geq \cdots k_n$ and r is a rational function with a zero in $\mathbb{D}$ and a pole in $\mathbb{C} \setminus \overline{\mathbb{D}}$. The factorization is said to be *canonical* if $D = I_n$, where I_n denotes the identity matrix. The above definition applies to functions belonging to other spaces (see *e.g.* [6]). In $[C_\mu]^{n\times n}$ G possesses a Wiener-Hopf factorization (1.3) *iff* G is invertible on S^1. We recall from [6] that the operator T_G is invertible *iff* the factorization (1.4) is canonical. This is the basic result from operatior theory that will be used to locate the singularities of the solution of the Lax equations. A direct consequence of Theorem 3.1 is Proposition 3.2 which

[1] We have omitted S^1 to simplify notation

states that the solution of equation (1.3) has a singularity at $t = t_i$ *iff* the Riemann-Hilbert problem

$$G\Phi^+ = \Phi^- \qquad \left(\Phi^+ \in \left[C_\mu^+\right]^n,\ \Phi^- \in \left[C_\mu^-\right]_0^n\right)$$

has a nontrivial solution at $t = t_i$. This is equivalent to saying that the Wiener-Hopf factorization of G is noncanonical for $t = t_i$.

The paper ends with an example of a dynamical system that belongs to the standard Lax class considered in [4]. For this example it is possible to obtain the solution to the Lax equation by classical methods (integration of the system of ordinary differential equations) and thus obtaining its set of singularities. This enables us to compare it with the set of singularities derived by our method. A rather interesting point is that the classical approach and the Lax equation one lead to different Riemann surfaces. The two surfaces are closely related, as shown in Proposition 4.2, but the fact that they are different led us to derive several intermediate results in order to show that the sets of singularities obtained by the two approaches coincide.

The study of the example given in Section 4 takes a large part of the paper but we believe that it not only illustrates the theory that we present here but also sheds some light into the relation between the classical methods and those based on the Lax equation - a point that may be obscure in the study of other finite-dimensional integrable systems, for example, in the study of some classical tops.

2. Lax equation and Riemann-Hilbert problems

In this section we generalize the results of [4, §2] replacing a neighborhood of the origin in the real variable t by a neighborhood of the origin in the now complex variable t (for what follows it is necessary to consider t as complex variable) and extending the class of equations considered. In [4] we studied a class of Lax equations of the form

$$\frac{dL_t}{dt} = \left[L_t^+, L_t\right] \tag{2.1}$$

where the dynamical variables L_t^+, L_t depend on a parameter λ varying on the unit circle S^1, L_t is a matrix-valued Laurent polynomial in λ and L_t^+ is the part of L_t analytic in the unit disc $\mathbb{D}$. In [4] we called the above class the *standard Lax class* (it includes *e.g.* a special case of the Lagrange top). In this paper we study a class of Lax equations more general than the above one. It includes most finite-dimensional integrable systems. We write the equations of this class in the form

$$\frac{dL_t}{dt} = \left[L_t^+ + A_0, L_t\right] \tag{2.2}$$

where L_t^+ is defined as above and $A_0 = P_0 L_t$, with P_0 being a bounded linear operator from the space $[C^1(\Omega)]^{n\times n}$ of matrix-valued Hölder functions on S^1 to the space $\mathbb{C}^{n\times n}$ of constant matrix functions on S^1 (depending on t as a parameter).

To state the first result in a rigorous way we need the definition that follows

Definition 2.1. *Let $[C^1(\Omega)]^{n\times n}$ be the space of continuously differentiable matrix functions with respect to t in a region $\Omega \subset \mathbb{C}$ and define $L_t(\lambda) \in [C^1(\Omega)]^{n\times n}$ to be a Laurent polynomial of the form*

$$L_t(\lambda) = \sum_{k=-m}^{1} L_t^{(k)} \lambda^k \qquad (m \in \mathbb{N}, \lambda \in S^1), \tag{2.3}$$

where $L_t^{(k)} \in \mathbb{C}^{n\times n}$. This gives for $L_t^+(\lambda)$ the expression

$$L_t^+(\lambda) = \sum_{k=0}^{1} L_t^{(k)} \lambda^k. \tag{2.4}$$

Remark 2.2. In the case of the standard Lax class ($A_0 = 0$ in (2.2)) equation (2.2) together with formulas (2.3) and (2.4) imply that $L_t^{(1)}$ is a constant of the dynamics.

We can now state our first result which is a generalization of [4, Theorem 2.3] extending the applicability of known formulas (see *e.g.* [9]) for L_t.

Theorem 2.3. *Let L_t be an $n \times n$ matrix-valued function satisfying the Lax equation (2.2) in a simply-connected region Ω containing the origin (in the variable t). Then L_t is given in the region Ω by the formulas*

$$L_t = \widetilde{G}_+ L_0 \widetilde{G}_+^{-1} = \widetilde{G}_-^{-1} L_0 \widetilde{G}_- \tag{2.5}$$

where $L_0 = L_t|_{t=0}$ and $\widetilde{G}_+, \widetilde{G}_-$ satisfy in Ω the linear differential equations

$$\frac{d\widetilde{G}_+}{dt} = (L_t^+ + A_0)\widetilde{G}_+ \qquad \frac{d\widetilde{G}_-}{dt} = \widetilde{G}_-(L_t^- - A_0) \tag{2.6}$$

subject to the initial conditions $\widetilde{G}_+|_{t=0} = \widetilde{G}_-|_{t=0} = I_n$, where I_n is the identity $n \times n$ matrix.

Proof. The proof goes along the same lines as the proof of [4, Theorem 2.3] with $L_t^\pm$ replaced by $L_t^\pm \pm A_0$. We note only that the condition on the connectivity of Ω is needed to ensure that equations (2.6) have well defined solutions throughout Ω. □

Proposition 2.4. *If the singularities of L_t with respect to t are isolated then there exists a simply connected region $\Omega = \mathbb{C} \setminus B_s$ where B_s is the union of two cuts joining the singularities.*

Proof. If the singularities are isolated we can denote them by t_n with $n \in \mathbb{Z}$. Furthermore we can enumerate them in *lexicographic order* ($t \leq t'$ *iff* $\operatorname{Re} t < \operatorname{Re} t'$ or $\operatorname{Re} t = \operatorname{Re} t'$ and $\operatorname{Im} t \leq \operatorname{Im} t'$). Consider two consecutive points of this sequence, say t_k, t_{k+1}. Define one cut (B_{s^+}) as the union of line segments $[t_{r-1}, t_r]$ for $r < k$. Similarly, define the second cut (B_{s^-}) as the union of segments $[t_r, t_{r+1}]$ for $r \geq k$. Then $\mathbb{C} \setminus (B_{s^+} \cup B_{s^-})$ is simply connected. □

Theorem 2.5. *Let* $G = \widetilde{G}_-\widetilde{G}_+$ *where* $\widetilde{G}_-, \widetilde{G}_+$ *satisfy equations* (2.6) *in a simply connected region* Ω *containing the origin (in the variable* t*) and the condition* $G|_{t=0} = I_n$*. Then*

$$G = \exp(tL_0). \tag{2.7}$$

Proof.

$$\begin{aligned} \frac{dG}{dt} &= \frac{d\widetilde{G}_-}{dt}\widetilde{G}_+ + \widetilde{G}_-\frac{d\widetilde{G}_+}{dt} \\ &= \widetilde{G}_-\left[\widetilde{G}_-^{-1}\frac{d\widetilde{G}_-}{dt} + \frac{d\widetilde{G}_+}{dt}\widetilde{G}_+^{-1}\right]\widetilde{G}_+. \end{aligned}$$

From equations (2.6)

$$\widetilde{G}_-^{-1}\frac{d\widetilde{G}_-}{dt} + \frac{d\widetilde{G}_+}{dt}\widetilde{G}_+^{-1} = L_t^- - A_0 + L_t^+ + A_0 = L_t.$$

Then

$$\frac{dG}{dt} = \widetilde{G}_-L_t\widetilde{G}_+ = L_0G,$$

where we have introduced the expression for L_0 resulting from the second of formulas (2.5), $L_0 = \widetilde{G}_-L_t\widetilde{G}_-^{-1}$. Formula (2.7) now follows from the above equation. □

Theorem 2.6.

(i) The factorization of $G = \exp(tL_0)$*,* $G = \widetilde{G}_-\widetilde{G}_+$ *is a canonical Wiener-Hopf factorization in the region* Ω *of Theorem 2.5.*

(ii) Let G_-G_+ *be another Wiener-Hopf factorization. Then*

$$\widetilde{G}_+ = G_+F_t, \qquad \widetilde{G}_- = G_-F_t^{-1}. \tag{2.8}$$

Proof. (i) Let $\widetilde{G}_-\widetilde{G}_+$ be the factorization of G obtained in Theorem 2.5, *i.e.*

$$G = \exp(tL_0) = \widetilde{G}_-\widetilde{G}_+.$$

This is a canonical Wiener-Hopf factorization of G in view of the properties of $\widetilde{G}_-, \widetilde{G}_+$ resulting from equations (2.6).

(ii) Let G_-G_+ be another (canonical) Wiener-Hopf factorization of G, obtained *e.g.* by solving a Riemann-Hilbert problem with coefficient G ($G\Phi^+ = \Phi^-$). Then we have

$$G = G_-G_+ = \widetilde{G}_-\widetilde{G}_+. \tag{2.9}$$

In the above relation both factorizations have factors that with their inverses are bounded analytic in their domains of existence, G_-, G_+, because it is assumed to be a Wiener-Hopf factorization and $\widetilde{G}_-$, $\widetilde{G}_+$ because the factors are assumed to satisfy equations (2.6).

From (2.9) we have

$$\widetilde{G}_-^{-1} G_- = \widetilde{G}_+ G_+^{-1} \tag{2.10}$$

which implies that both sides equal a constant in λ, but since we have a relation (2.10) for each t, both sides must equal a function of t, independent of λ. We write

$$\widetilde{G}_-^{-1} G_- = \widetilde{G}_+ G_+^{-1} = F_t$$

i.e.

$$\widetilde{G}_- = G_- F_t^{-1}, \qquad \widetilde{G}_+ = F_t G_+.$$

□

F_t plays the role of a normalization factor at the point t for G_-, G_+.

Theorem 2.7. *Let F_t satisfy the linear differential equation*

$$\frac{dF_t}{dt} = A_0 F_t. \tag{2.11}$$

Then the function L_t given by

$$L_t = F_t \hat{L}_t F_t^{-1}, \tag{2.12}$$

with $\hat{L}_t = G_+ L_0 G_+^{-1}$, satisfies the Lax equation (2.2)*. Here G_+ is as in Theorem 2.6 (as are G_-, G).*

Proof. We show that L_t given by (2.12) with F_t satisfying (2.11) is a solution to equation (2.2). We have

$$\begin{aligned} \frac{dL_t}{dt} &= A_0 F_t \hat{L}_t F_t^{-1} + F_t \frac{d\hat{L}_t}{dt} F_t^{-1} - F_t \hat{L}_t F_t^{-1} A_0 F_t F_t^{-1} \\ &= [A_0, L_t] + F_t \frac{d\hat{L}_t}{dt} F_t^{-1}. \end{aligned}$$

Since $\hat{L}_t = G_+ L_0 G_+^{-1}$ it satisfies a standard Lax equation

$$\frac{d\hat{L}_t}{dt} = \left[\hat{L}_t^+, \hat{L}_t\right]$$

(see *e.g.* [4, proof of Theorem 2.7]). Hence, noting that $L_t = F_t \hat{L}_t F_t^{-1}$, we get

$$\frac{dL_t}{dt} = \left[L_t^+ + A_0, L_t\right].$$

which is equation (2.2). □

Proposition 2.8. *Equation* (2.11) *is equivalent to the equation in* Ω

$$\frac{dF_t}{dt} = F_t \hat{A}_0 \tag{2.13}$$

where $\hat{A}_0 = P_0 \hat{L}_t = P_0 G_+ L_0 G_+^{-1}$.

Proof. Let $A_0 = P_0 L_t$ where P_0 is a linear bounded operator as assumed in the definition of the right-hand side of equation (2.2). Define

$$\hat{A}_0 = F_t^{-1} A_0 F_t. \tag{2.14}$$

Noting that F_t is independent of λ, F_t commutes with P_0 which leads to

$$\hat{A}_0 = P_0 F_t^{-1} L_t F_t = P_0 \hat{L}_t = P_0 G_+ L_0 G_+^{-1}.$$

Introducing in equation (2.11) the definition (2.14) of $\hat{A}_0$ we obtain

$$\frac{dF_t}{dt} = F_t \hat{A}_0 F_t^{-1} F_t = F_t \hat{A}_0,$$

as required. □

Remark 2.9. Equation (2.13) is more convenient for calculating the solution of equation (2.2) since $\hat{A}_0$ is known explicitly whereas A_0 is not.

Proposition 2.10. $G = \exp(tL_0)$ *has a canonical factorization at a point* $t_i \in \Omega$ *with factors differentiable* w.r.t. t *in a neighborhood of* t_i iff *equation* (2.2) *has a solution at the point* t_i.

Proof. Sufficiency: Assume that equation (2.2) has a solution at a point $t_i \in \Omega$. Then by Theorem 2.3 there exist functions $\widetilde{G}_-$, $\widetilde{G}_+$ satisfying (2.6) in a neighborhood of t_i (Ω is open) which give the solution to equation (2.2),

$$L_t = \widetilde{G}_+ L_0 \widetilde{G}_+^{-1} = \widetilde{G}_-^{-1} L_0 \widetilde{G}_-.$$

From Theorem 2.6 $\widetilde{G}_+$, $\widetilde{G}_-$ are related to the factors of another canonical factorization of G (G_-, G_+) by the formulas

$$G_- = \widetilde{G}_- F_t, \qquad\qquad G_+ = F_t^{-1} \widetilde{G}_+$$

where F_t satisfies the differential equation (2.11). Since $\widetilde{G}_-$, $\widetilde{G}_+$ and F_t are differentiable in a vicinity of t_i it follows that the factors G_-, G_+ are differentiable too.

Necessity: Assume that the factors G_-, G_+ of a canonical factorization of G are differentiable. Then

$$\frac{dG}{dt} = L_0 G = \frac{dG_-}{dt} G_+ + G_- \frac{dG_+}{dt}$$

and letting $\hat{L}_t = G_-^{-1} L_0 G_-$ we get from the above relation

$$\hat{L}_t^+ := P^+ \hat{L}_t = \frac{dG_+}{dt} G_+^{-1},$$

and, putting $L_t = F_t \hat{L}_t F_t^{-1}$ with F_t satisfying (2.11), we have (see the proof of Theorem 2.7)

$$\frac{dL_t}{dt} = \left[L_t^+ + A_0, L_t\right]$$

which is equation (2.2). □

3. Singularities via the Riemann-Hilbert approach

In this section we present our main result which enables us to locate the singularities of the solution to equation (2.2) without obtaining the explicit solution of the associated Riemann-Hilbert problem. Here the use of the factorization method is crucial since it allows us to translate the problem of the existence of singularities into an operator theory problem.

We recall from the Introduction the direct sum decomposition of $[C_\mu(S^1)]^n$,

$$\left[C_\mu\left(S^1\right)\right]^n = \left[C_\mu^+\right]^n \oplus \left[C_\mu^-\right]_0^n, \tag{3.1}$$

where $\left[C_\mu^+\right]^n$ denotes the subspace of $[C_\mu(S^1)]^n$ of functions analytic in $\mathbb{D}$ and $\left[C_\mu^-\right]_0^n$ is the subspace of analytic functions in $\mathbb{C}\setminus\overline{\mathbb{D}}$ that vanish at infinity. We let $P^+\colon [C_\mu(S^1)]^n \to \left[C_\mu^+\right]^n$ denote the projection associated to this decomposition (so that $\ker P^+ = \left[C_\mu^-\right]_0^n$).

Given a matrix $G \in [C_\mu(S^1)]^{n\times n}$ the corresponding multiplication operator in $\left[C_\mu^+\right]^n$ is denoted GI_+. The composite

$$P^+GI_+\colon \left[C_\mu^+\right]^n \to \left[C_\mu^+\right]^n$$

is a Topelitz operator (with symbol G [6, Ch.1]) whose properties are closely related to those of its symbol G. We recall from [6, Ch.1] that the operator P^+GI_+ is invertible *iff* G has a canonical Wiener-Hopf factorization

$$G = G_-G_+,$$

with $(G^\pm)^{\pm 1} \in \left[C_\mu^\pm(S^1)\right]$.

We are now ready to state the main result of this section.

Theorem 3.1. *Let T_G be the Toeplitz operator P^+GI_+ defined above, where $G = \exp(tL_0)$.*

Then equation (2.2) *has a solution in a neighborhood of a point $t = t_i$* iff *the operator T_G is injective at that point,* i.e., $\ker T_G$ *is trivial.*

Proof. Sufficiency: We begin by proving that if $\ker T_G$ is trivial T_G is invertible. Firstly we note that $G = \exp(tL_0) \in C_\mu(S^1)$ (in fact $G \in C^\infty(S^1)$) for every $t \in \mathbb{C}$. Thus a factorization of G of the general form

$$G = G_-DG_+, \qquad (G_\pm)^{\pm 1} \in \left[C_\mu^\pm\left(S^1\right)\right]^{n\times n},$$

where D is a diagonal nonsingular rational matrix, exists for all $t \in \mathbb{C}$ (see *e.g.* [6, Ch.1]).

Since $G \in [C_\mu(S^1)]^{n\times n}$, $\det G \in C_\mu(S^1)$ and

$$\det G = \exp((\operatorname{tr} L_0)t) \neq 0,$$

for all $\lambda \in S^1$, it follows that T_G is Fredholm of zero index. This means that

$$\operatorname{codim} \operatorname{im} T_G = \dim \ker T_G.$$

It follows that, if $\ker T_G$ is trivial at $t = t_i$, T_G is invertible at this point. It is easy to see that this is true in a neighborhood of t_i. Hence G has a canonical factorization

$$G = \exp(tL_0) = G_- G_+$$

in a neighborhood of t_i. By Proposition 2.10 this implies that equation (2.2) has a solution at this point.

Necessity: Assume that a solution to equation (2.2) exists at $t = t_i$. Then, by Proposition 2.10, G possesses a canonical factorization at $t = t_i$, *i.e.*,

$$G = G_- G_+, \qquad (G_\pm)^{\pm 1} \in \left[C_\mu^\pm\left(S^1\right)\right]^{n\times n}.$$

This is equivalent to the invertibility of T_G in a neighborhood of t_i and thus $\ker T_G$ is trivial. □

In the next two propositions we express the condition of Theorem 3.1 in terms of the existence of solutions to a certain Riemann-Hilbert problem, which has the advantage of being easier to analyse.

Proposition 3.2. *Let T_G be the operator defined in Theorem 3.1. Then* $\ker T_G$ *is nontrivial* iff *the Riemann-Hilbert problem*

$$G\Phi^+ = \Phi^-, \qquad \Phi^\pm \in \left[C_\mu^\pm(S^1)\right]^n,$$

with $\Phi^-(\infty) = 0$, has non-trivial solutions.

Proof. $\ker T_G$ being non-trivial means that the equation

$$P^+ G\Phi^+ = 0, \qquad \Phi^+ \in \left[C_\mu^+\right]^n$$

has non-trivial solutions. Keeping in mind the direct sum decomposition (3.1), we see that this is equivalent to saying that the Riemann-Hilbert problem in $[C_\mu(S^1)]^n$

$$G\Phi^+ = \Phi^- \quad \text{with} \quad \Phi^-(\infty) = 0,$$

has non-trivial solutions. □

Proposition 3.3. *Let $n = 2$ in Proposition 3.2. Then the vector valued Riemann-Hilbert problem (on the Riemann sphere)*

$$G\Phi^+ = \Phi^-, \quad \Phi^-(\infty) = 0, \tag{3.2}$$

given in Proposition 3.2 is equivalent to a scalar Riemann-Hilbert problem of the form

$$g\Psi^+ = \Psi^- \tag{3.3}$$

on a compact Riemann surface $\varSigma$ defined by the equation $\det(\mu I_2 - L(\lambda)) = 0$ *with Ψ^- subject to the condition*

$$\Psi^-(\infty_1) = 0,\ \Psi(\infty_2) = 0 \tag{3.4}$$

where ∞_1, ∞_2 are the poles of the meromorphic function given by the projection

$$\varSigma \to \mathbb{P}^1(\mathbb{C}), (x, w) \mapsto x$$

*(*i.e., *∞_1, ∞_2 are the points of $\varSigma$ "at infinity").*

Proof. It is proven in [5] that the Riemann-Hilbert problem (3.2) is equivalent, for $n = 2$, to a scalar Riemann-Hilbert problem on $\varSigma$ (3.3). The condition (3.4) is the translation of the condition $\Phi^-(\infty) = 0$ in (3.2) to the Riemann surface. □

4. Example

In this section we study a dynamical system for which the solution and, consequently, its singularities can be obtained by classical methods and compare the result obtained with that given by the method of Section 3.

4.1. **Dynamical system.** We take the example presented in [4] which is given by the equations

$$\frac{dL_t}{dt} = \left[L_t^+, L_t\right] \tag{4.1}$$

where

$$L_t(\lambda) = \begin{bmatrix} v(\lambda) & u(\lambda) \\ w(\lambda) & -v(\lambda) \end{bmatrix}, \qquad \lambda \in S^1 \tag{4.2}$$

with

$$\begin{aligned} v(\lambda) &= z\lambda^{-1} \\ u(\lambda) &= a\lambda + y\lambda^{-1} + x, \quad a \in \mathbb{C} \\ w(\lambda) &= a\lambda + y\lambda^{-1} - x \end{aligned} \tag{4.3}$$

and L_t^+ being the polynomial part of L_t (with respect to λ). It can easily be seen that equation (4.1) together with (4.2) and (4.3) is equivalent to the following nonlinear system of differential equations

$$\frac{dx}{dt} = -2az, \qquad \frac{dy}{dt} = -2xz, \qquad \frac{dz}{dt} = 2xy \tag{4.4}$$

for the dynamical variables x, y, z. This system admits two integrals of the motion, namely,

$$A = x^2 - 2ay, \qquad\qquad B = y^2 + z^2. \tag{4.5}$$

That these are invariants is easily checked by differentiating both sides of relations (4.5) and using equations (4.4).

4.2. **Classical solution.** To obtain an equation of the movement in the variable x we begin with the first of equations (4.4)

$$(\dot{x})^2 = 4a^2z^2 = 4a^2(B - y^2)$$

where $\dot{x} = \frac{dx}{dt}$. Using relations (4.5) yields

$$(\dot{x})^2 = 4a^2B - (A - x^2)^2 = 4a^2B - A^2 + 2Ax^2 - x^4$$

from which we get

$$\dot{x} = \mathrm{i}\sqrt{p(x)} \tag{4.6}$$

where $p(x) = x^4 - 2Ax^2 + A^2 - 4a^2B$. The above equation means that $(\dot{x}, x)$ lies in an elliptic curve, *i.e.*, the orbits of the dynamics lie in an elliptic Riemann surface.

Before we integrate (4.6) we note that if we derive equations for the variables y, z we obtain equation (4.6) after an elementary transformation on these variables as was to be expected.

Integration of (4.6) gives

$$\int_{x_0}^{x} \frac{dx}{\sqrt{p(x)}} = \mathrm{i}t \tag{4.7}$$

where x_0 is the value of x at $t = 0$, and the path of integration is understood to be on the Riemann surface Σ defined by

$$w^2 = p(x) = (x^2 - x_1^2)(x^2 - x_2^2) \tag{4.8}$$

with the zeros of $p(x)$, $\pm x_1$, $\pm x_2$, given by

$$x_1^2 = A + 2a\sqrt{B}, \qquad x_2^2 = A - 2a\sqrt{B}. \tag{4.9}$$

It is useful to write (4.8) in the normalized form

$$w^2 = x_1^2x_2^2(1 - \widetilde{x}^2)(1 - k^2\widetilde{x}^2)$$

where $\widetilde{x} = x/x_1$ and $k^2 = x_1^2/x_2^2$. From now on we take as a definition of the Riemann surface Σ the following equation

$$w^2 = \left(1 - x^2\right)\left(1 - k^2x^2\right), \tag{4.10}$$

which corresponds to making the change of variables $x \mapsto x/x_1$, $w \mapsto w/(x_1x_2)$. With this notation, (4.7) takes the form

$$\int_{\widetilde{x}_0}^{\widetilde{x}} \frac{dx}{w(x)} = \mathrm{i}x_2t \qquad \left(\text{on } \Sigma\ ^2\right)$$

with $\widetilde{x} = x/x_1$, $\widetilde{x}_0 = x_0/x_1$. It is convenient to write the above integral as a difference of two integrals as follows

$$\mathrm{i}x_2 t = \int_0^{\widetilde{x}} \frac{dx}{w(x)} - \int_0^{\widetilde{x}_0} \frac{dx}{w(x)} \qquad (\text{on } \Sigma) \tag{4.11}$$

We are looking for the singularities of the solution to equation (4.6) so we let $x \to \infty$ which leads to

$$\int_0^{\widetilde{x}} \frac{dx}{w(x)} \to \mathrm{i}\mathbf{K}' \qquad \text{or} \qquad \int_0^{\widetilde{x}} \frac{dx}{w(x)} \to \mathrm{i}\mathbf{K}' + 2\mathbf{K},$$

where $\mathbf{K}$, $\mathbf{K}'$ are, respectively, the complete elliptic integral and the complementary complete elliptic integral of the first kind (see *e.g.* [2]). Hence from (4.11), keeping in mind that (4.11) is an equation on Σ, we obtain

$$\mathrm{i}x_2 t = \mathrm{i}\mathbf{K}' - u_0 + 4m\mathbf{K} + 2\mathrm{i}\mathbf{K}',$$

or

$$\mathrm{i}x_2 t = \mathrm{i}\mathbf{K}' + 2\mathbf{K} - u_0 + 4m\mathbf{K} + 2\mathrm{i}n\mathbf{K}', \qquad n, m \in \mathbb{Z},$$

where

$$u_0 = \int_0^{\widetilde{x}_0} \frac{dx}{w(x)}. \tag{4.12}$$

The above formulas for t are equivalent to the single formula

$$\mathrm{i}x_2 t = -u_0 + \mathrm{i}\mathbf{K}' + 2m\mathbf{K} + 2\mathrm{i}n\mathbf{K}', \qquad n, m \in \mathbb{Z}. \tag{4.13}$$

This relation gives us the values of t at which singularities occur, *i.e.*, where the solution blows up.

4.3. **Riemann-Hilbert solution.** Next we derive a formula for the singularities of the solution to system (4.4) using the method of Propositions 3.2 and 3.3. To this end we need to formulate the Riemann-Hilbert problem (3.2) for the function $G = \exp(tL_0)$ in an associated Riemann surface. Recalling (3.2) we have

$$G\Phi^+ = \Phi^- \qquad \text{with} \qquad G = \exp(tL_0), \tag{4.14}$$

and $\Phi^\pm \in \left[C_\mu^\pm\right]^2$ with the condition $\Phi^-(\infty) = 0$.

Taking into account that L_0 can be diagonalized as

$$L_0 = S D_0 S^{-1}, \tag{4.15}$$

with

$$S = \begin{bmatrix} 1 & -1 \\ \frac{\mu - z_0}{q_1(\lambda)} & \frac{\mu + z_0}{q_1(\lambda)} \end{bmatrix} \qquad \left(q_1(\lambda) := a\lambda^2 + x_0\lambda + y_0 \right) \tag{4.16}$$

[2]We identity Σ with the quotient of $\mathbb{C}$ by the lattice of periods of $dx/\sqrt{p(x)}$.

and $D_0 = \mathrm{diag}(\lambda^{-1}\mu, -\lambda^{-1}\mu)$ where $\mu = \lambda\nu$ with ν given by the characteristic equation of L_0,

$$\det\left(\nu I_2 - L_0(\lambda)\right) = 0. \tag{4.17}$$

From this equation we obtain

$$\mu^2 = p_1(\lambda) := a^2\lambda^4 - (x_0^2 - 2ay_0)\lambda^2 + z_0^2 + y_0^2 \tag{4.18}$$

or, introducing the invariants A and B,

$$p_1(\lambda) = a^2\lambda^4 - A\lambda^2 + B. \tag{4.19}$$

The explicit formulas for the zeros of $p_1(\lambda)$, $\pm\lambda_1, \pm\lambda_2$, are given in (4.33) below. Relation (4.18) defines an elliptic Riemann surface, which is associated with L_0 (or L_t as it is independent of the dynamics). We denote by Σ_1 the *compact Riemann surface* obtained by adding two points at infinity ∞_1, ∞_2.

Going back to (4.15) it follows from it that

$$G = \exp\left(tL_0\right) = SDS^{-1}$$

where $D = \mathrm{diag}(\exp(t\lambda^{-1}\mu), \exp(-t\lambda^{-1}\mu))$.

Hence (4.14) may be written as

$$DS^{-1}\Phi^+ = S^{-1}\Phi^-,$$

which, in terms of the components of $\Phi^\pm$, denoted $(\phi_1^\pm, \phi_2^\pm)$, is written as

$$\begin{cases} d_1\left(z_0\phi_1^+ + q_1\phi_2^+ + \mu\phi_1^+\right) & = z_0\phi_1^- + q_1\phi_2^- + \mu\phi_1^- \\ d_2\left(z_0\phi_1^+ + q_1\phi_2^+ - \mu\phi_1^+\right) & = z_0\phi_1^- + q_1\phi_2^- - \mu\phi_1^- \end{cases} \tag{4.20}$$

where $d_1 = \exp(t\lambda^{-1}\mu)$, $d_2 = \exp(-t\lambda^{-1}\mu)$.

The above system is equivalent to the following single scalar equation (for more details see [4] or [5]) on a contour Γ that is the preimage of S^1 under the projection $\varrho\colon (\lambda,\mu) \mapsto \lambda$,

$$d\left(\phi_2^+ + \frac{z_0+\mu}{q_1}\phi_1^+\right) = \phi_2^- + \frac{z_0+\mu}{q_1}\phi_1^-. \tag{4.21}$$

Note that Γ has two connected components; we put $d = d_1$ on one of these components and $d = d_2$ on the other. In view of the expressions for d_1 and d_2 we have

$$d = \exp\left(\frac{\mu}{\lambda}t\right), \qquad (\lambda,\mu) \in \Gamma. \tag{4.22}$$

Concerning equation (4.21), it is also useful to note that setting $q_2(\lambda) = a\lambda_2 - x_0\lambda + y_0$, we have

$$\mu^2 - z_0^2 = q_1(\lambda)q_2(\lambda). \tag{4.23}$$

It follows that, as a meromorphic function on Σ_1, $q_1(\lambda)$ has four zeros, two of which are zeros of $\mu + z_0$ and the other two are zeros of $\mu - z_0$.

To solve (4.21) we note that d can be factorized on the Riemann surface as

$$d = d_- r d_+,$$

where $(d^+)^{\pm 1} \in C_\mu(\Gamma)$ extends holomorphically to the preimage Ω^+ of $\mathbb{D}$ under the projection ϱ and, similarly, $(d^-)^{\pm 1} \in C_\mu(\Gamma)$ admits a holomorphic extension to the preimage Ω^- of $\mathbb{P}(\mathbb{C}) \setminus \overline{\mathbb{D}}$. Finally, r is a rational function on Σ_1. See [4] or [5] for more details.

Note that all three factors in the above factorization depend on t. Introducing this factorization in (4.21), we get

$$rd_+ \left(\phi_2^+ + \frac{z_0 + \mu}{q_1} \phi_1^+ \right) = d_-^{-1} \left(\phi_2^- + \frac{z_0 + \mu}{q_1} \phi_1^- \right) = R, \tag{4.24}$$

where R is a rational function on Σ_1.

For the computations that follow it is convenient to rewrite the Riemann surface Σ_1 using the normalized equation

$$\mu^2 = (1 - \lambda^2)(1 - k_1^2 \lambda^2), \tag{4.25}$$

where $k_1 = \lambda_1/\lambda_2$ and $\pm\lambda_1, \pm\lambda_2$ are the roots of $p_1(\lambda)$ (*cf.* (4.18)) given in (4.33) below. This corresponds to making a change of variables $\lambda \mapsto \lambda/\lambda_1$, $\mu \mapsto \mu/(a\lambda_1\lambda_2)$.

Also, from now on we identify Σ_1 with its *Jacobian*, using the Abel map

$$(\lambda, \mu) \mapsto u = \int_0^{(\lambda,\mu)} \frac{d\lambda}{\mu}, \tag{4.26}$$

i.e., we consider all equations relating points of Σ_1 as written on the quotient of $\mathbb{C}$ by the lattice of periods of the holomorphic form $d\lambda/\mu$.

Hence, keeping in mind that Σ_1 is an elliptic Riemann surface (p_1 is a fourth degree polynomial), R can be expressed in elliptic theta functions. To this end we recall that we are solving (4.24) with the conditions $\phi_i^-(\infty_j) = 0$, for $i, j = 1, 2$, where ∞_1, ∞_2 are the two points at infinity [3] in Σ_1, which correspond to ∞ under the projection $\varrho\colon (\lambda, \mu) \mapsto \lambda$. Denoting by ψ^+, ψ^- the expression within parentheses in both sides of (4.24), these conditions correspond to

$$\psi^-(\infty_1) = 0, \qquad \psi^-(\infty_2) = 0. \tag{4.27}$$

Before introducing these conditions we note that, using the Jacobi theta function ϑ_1 that satisfies $\vartheta_1(0) = 0$, R has the expression

$$R(u) = \gamma \frac{\vartheta_1(u - v_0)\vartheta_1(u - v_1)\vartheta_1(u - v_2)}{\vartheta_1(u - u_0)\vartheta_1(u - u_1)\vartheta_1(u - u_2)} \tag{4.28}$$

where $\gamma \in \mathbb{C}$ and the zeros and poles of R are determined by the following conditions:

[3] We choose ∞_1 such that $\mu \sim k_1 \lambda^2$ near ∞_1.

(i) R has a pole at the point u_0 corresponding to the pole of r in Ω^+ (see [4, Appendix B]);

(ii) R has a zero at the point v_0 corresponding to the zero of r in Ω^+ (see [4, Appendix B]);

(iii) R has two poles u_1, u_2 at the zeros of q_1 that do not coincide with the zeros of $a\lambda_1\lambda_2\mu + z_0$ (which is $\mu + z_0$ in (4.24) written in the normalized coordinates of (4.25));

(iv) R has two zeros at points v_1, v_2 imposed by condition (4.27), *i.e.*,

$$v_1 = \infty_1, \quad v_2 = \infty_2.$$

(v) The zeros and poles of R must satisfy Abel's condition:

$$v_0 - u_0 = u_1 + u_2 - \infty_1 - \infty_2 \qquad (\mathrm{mod}\, 2\mathrm{i}\mathbf{K}_1' + 4\mathbf{K}_1), \tag{4.29}$$

where $\mathbf{K}_1$ and $\mathbf{K}_1'$ are the complete elliptic and complementary elliptic integrals of the first kind of Σ_1.

From the analysis of the factorization of the function d given in (4.22) (see [4, Definition B.7 and Proposition B.9]) we obtain

$$v_0 - u_0 = 2at\lambda_2 \tag{4.30}$$

where λ_2 is as in the expression for $p_1(\lambda)$ (see text following (4.19)). Substitution of (4.30) in (4.29) gives us the expression for the values of t for which singularities occur. Taking into account that $\infty_1 + \infty_2 = 2\mathbf{K}_1 + 2\mathrm{i}\mathbf{K}_1'$ (mod $4m\mathbf{K}_1 + 2\mathrm{i}n\mathbf{K}_1'$) we have

$$2at\lambda_2 = u_1 + u_2 + 2\mathbf{K}_1 + 4m\mathbf{K}_1 + 2\mathrm{i}n\mathbf{K}_1' \tag{4.31}$$

where u_1, u_2 are the images under Abel's map of the zeros of q_1 that do not coincide with zeros of $a\lambda_1\lambda_2\mu + z_0$, *i.e.*,

$$u_1 = \int_0^{\hat\lambda_1/\lambda_1} \frac{d\lambda}{\mu}, \qquad\qquad u_2 = \int_0^{\hat\lambda_2/\lambda_1} \frac{d\lambda}{\mu},$$

where $\hat\lambda_1, \hat\lambda_2$ are de zeros of $q_1(\lambda)$ and λ_1 is a zero of $p_1(\lambda)$ given in (4.33) below.

Thus (4.31) gives us the values of t leading to singularities of the solution of Lax equation (2.2) as derived from the theory of Section 3.

Remark 4.1. (i) Formula (4.31) was obtained without requiring an explicit formula for $\phi_1^\pm$, $\phi_2^\pm$ corresponding to the factors of the canonical factorization of G, $G = G_- G_+$, although these functions can easily be obtained from (4.24), replacing condition *(iv)* by the imposition of a zero at a chosen point v_1. Then Abel's condition *(v)* gives the zero v_2 (see [4] for the details). The solution thus obtained gives the factors G_-, G_+ of G providing t does not satisfy (4.31), a result that could not be obtained in [4].

(ii) Formulas (4.31) and (4.13) are not easily compared since they involve different Riemann surfaces. The appearance of distinct

surfaces when using different methods to study integrable systems is an intriguing phenomenon that occurs in other examples [3].

We show next that the two Riemann surfaces are closely related and that the two expressions for the singularities coincide.

4.4. **Comparison of solutions.** We start by showing that the two Riemann surfaces Σ and Σ_1 are related and this will enable us to express both formulas (4.13) and (4.31) on the same Riemann surface thus allowing for a comparison of the two results.

The Riemann surface Σ is defined by the equation

$$w^2 = (1 - x^2)(1 - k^2x^2) = \frac{p(x_1x)}{(x_1x_2)^2}$$

with $p(x) = x^4 - 2Ax^2 + A^2 - 4a^2B = (x^2 - x_1^2)(x^2 - x_2^2)$ and $k = x_1/x_2$, where

$$x_1^2 = A + 2a\sqrt{B}, \qquad x_2^2 = A - 2a\sqrt{B}. \tag{4.32}$$

The Riemann surface Σ_1 is defined by the equation

$$\mu^2 = (1 - \lambda^2)(1 - k_1^2\lambda^2) = \frac{p_1(\lambda_1\lambda)}{(a\lambda_1\lambda_2)^2}$$

with $p_1(\lambda) = a^2\lambda^4 - A\lambda^2 + B = a^2(\lambda^2 - \lambda_1^2)(\lambda^2 - \lambda_2^2)$, where

$$\lambda_1^2 = \frac{A + \sqrt{A^2 - 4a^2B}}{2a^2}, \qquad \lambda_2^2 = \frac{A - \sqrt{A^2 - 4a^2B}}{2a^2}. \tag{4.33}$$

From the expression for $p(x)$ and (4.32) we have

$$A^2 - 4a^2B = x_1^2x_2^2$$
$$2A = x_1^2 + x_2^2.$$

Introducing these relations in (4.33) gives

$$\lambda_1^2 = \left(\frac{x_2 - x_1}{2a}\right)^2, \qquad \lambda_2^2 = \left(\frac{x_1 + x_2}{2a}\right)^2 \tag{4.34}$$

which leads to

$$\lambda_1 = \frac{x_2 - x_1}{2a}, \qquad \lambda_2 = \frac{x_1 + x_2}{2a} \tag{4.35}$$

where the sign in the square root is determined by a direct check on (4.33). From (4.35) we now get the relation between the moduli of the two surfaces

$$k_1 = \frac{\lambda_1}{\lambda_2} = \frac{x_2 - x_1}{x_2 + x_1} = \frac{1 - k}{1 + k} \tag{4.36}$$

where k and k_1 denote the *elliptic moduli* of Σ and Σ_1, respectively (see [2]). This shows that the surfaces are closely related as claimed at the end of Section 4.3.

Having obtained equality (4.36) we are now in a position to state the following proposition relating Σ and Σ_1.

Proposition 4.2. *The following statements express the relation between the Riemann surfaces Σ and Σ_1:*

(i) For the elliptic moduli of Σ and Σ_1, respectively k, k_1, we have

$$k_1 = \frac{1-k}{1+k}$$

(ii) There is a holomorphic map $\varphi\colon \Sigma \to \Sigma_1$ given by

$$\Sigma \ni (x,w) \mapsto (\lambda,\mu) := \left(\mathrm{i}(1+k)\frac{x}{w}, \frac{k^2x^4-1}{w^2}\right) \in \Sigma_1.$$

(iii) Under the map φ of (ii) the points at infinity of Σ are mapped to $\mathbf{o}_1 := (0,1)$ and $(0,\pm 1)$, $(\pm 1, 0)$, $(\pm 1/k, 0)$ are mapped to $\mathbf{o}_2 := (0,-1)$, ∞_1 and ∞_2, respectively.

(iv) The relation between the holomorphic forms of both surfaces is expressed by

$$\varphi^*\left(\frac{d\lambda}{\mu}\right) = -\mathrm{i}(1+k)\frac{dx}{w}.$$

Proof. (i) was proven in (4.36).

The formula in *(ii)* is obtained by composing the two Gauss transformations corresponding in terms of elliptic moduli to $k \mapsto k_1'$ and $k' \mapsto k_1$ (see [2, §39]). That it defines a map $\Sigma \to \Sigma_1$ can be directly checked by a substitution of (4.36) in $\mu^2 = (1-\lambda^2)(1-k_1^2\lambda^2)$. We note that this map is not injective; in fact it is 2 to 1. *(iii)* is easily obtained by direct substitution in formula *(ii)*.

The expression *(iv)* follows directly from *(ii)* by differentiation. □

Before we attempt to formulate expression (4.13) in Σ_1 we are going to write $u_1 + u_2$ of (4.31) as a single integral as in (4.13) in order to make it possible to compare the two results. From (4.31)

$$u_1 + u_2 = \int_0^{\hat\lambda_1/\lambda_1} \frac{d\lambda}{\mu} + \int_0^{\hat\lambda_2/\lambda_1} \frac{d\lambda}{\mu}$$

which we seek to write in the form

$$u_1 + u_2 = \int_0^{\xi_0} \frac{d\lambda}{\mu}. \tag{4.37}$$

To obtain ξ_0 we make use of the formula for the sum of arguments of the elliptic function sn (see [2]),

$$\operatorname{sn}(u_1+u_2) = \frac{\operatorname{sn} u_1 \operatorname{cn} u_2 \operatorname{dn} u_2 + \operatorname{sn} u_2 \operatorname{cn} u_1 \operatorname{dn} u_1}{1 - k_1^2 \operatorname{sn}^2 u_1 \operatorname{sn}^2 u_2}. \tag{4.38}$$

We have:

$$\operatorname{sn} u_1 = \frac{\hat{\lambda}_1}{\lambda_1}, \qquad \operatorname{sn} u_2 = \frac{\hat{\lambda}_2}{\lambda_1},$$

$$\operatorname{cn} u_2 \operatorname{dn} u_2 = \left[1 - \left(\frac{\hat{\lambda}_2}{\lambda_1}\right)^2\right]^{1/2} \cdot \left[1 - k_1^2 \left(\frac{\hat{\lambda}_2}{\lambda_1}\right)^2\right]^{1/2} = \mu\left(\frac{\hat{\lambda}_2}{\lambda_1}\right),$$

$$\operatorname{cn} u_1 \operatorname{dn} u_1 = \mu\left(\frac{\hat{\lambda}_1}{\lambda_1}\right).$$

Since $q_1(\lambda_1\lambda)q_2(\lambda_1\lambda) = (a\lambda_1\lambda_2\mu)^2 - z_0^2$ (see (4.23)) and $\hat{\lambda}_1$, $\hat{\lambda}_2$ are the zeros of q_1 that are not zeros of $a\lambda_1\lambda_2 + z_0$, we have

$$\mu\left(\frac{\hat{\lambda}_1}{\lambda_1}\right) = \mu\left(\frac{\hat{\lambda}_2}{\lambda_1}\right) = \frac{z_0}{a\lambda_1\lambda_2} \tag{4.39}$$

where the factor $1/(a\lambda_1\lambda_2)$ comes from the normalization of μ.

Using the above results in (4.38), taking into account, (4.37) gives

$$\xi_0 = \frac{1}{\lambda_1}\,\frac{\hat{\lambda}_1 + \hat{\lambda}_2}{1 - k_1^2\hat{\lambda}_1^2\hat{\lambda}_2^2/\lambda_1^4}\,\frac{z_0}{a\lambda_1\lambda_2} \tag{4.40}$$

The denominator of the above formula can be simplified as follows

$$1 - k_1^2\,\frac{\hat{\lambda}_1^2\hat{\lambda}_2^2}{\lambda_1^4} = 1 - \frac{\hat{\lambda}_1^2\hat{\lambda}_2^2}{\lambda_1^2\lambda_2^2}$$

since $k_1^2 = \lambda_1^2/\lambda_2^2$. Using (4.39) we obtain

$$1 - \frac{\hat{\lambda}_1^2\hat{\lambda}_2^2}{\lambda_1^2\lambda_2^2} = 1 - \frac{y_0^2}{a^2\lambda_1^2\lambda_2^2} = \frac{z_0^2}{B}$$

as $\hat{\lambda}_1^2\hat{\lambda}_2^2 = y_0^2/a^2$, $\lambda_1^2\lambda_2^2 = B/a^2$ and $B = y_0^2 + z_0^2$.

Finally,

$$\xi_0 = \frac{x_0}{\lambda_1 a}\,\frac{z_0}{a\lambda_1\lambda_2}\,\frac{B}{z_0^2} = \frac{x_0}{z_0}\lambda_2 \tag{4.41}$$

where we have used the result $\hat{\lambda}_1 + \hat{\lambda}_2 = x_0/a$.

We shall now transform the terms on the right-hand side of (4.13) into the surface Σ_1. We first take the expression for u_0 given in (4.12)

$$u_0 = \int_0^{x_0/x_1} \frac{dx}{w(x)} = \frac{\mathrm{i}}{1+k}\int_0^{\lambda_0} \frac{d\lambda}{\mu(\lambda)}, \tag{4.42}$$

where

$$\lambda_0 = \mathrm{i}(1+k)\,\frac{x_0/x_1}{w(x_0/x_1)}. \tag{4.43}$$

For the sake of simplicity in the calculations instead of trying to transform λ_0 into ξ_0 we prefer to take ξ_0 in (4.41) and transform it as follows:

$$\begin{aligned} \xi_0 &= \frac{x_0}{z_0}\,\lambda_2 = \frac{x_0}{w(x_0/x_1)}\,\frac{2\mathrm{i}a}{x_1x_2}\,\lambda_2 \\ &= \frac{x_0}{w(x_0/x_1)}\,\frac{2\mathrm{i}a}{x_1x_2}\,\frac{x_1+x_2}{2a} \\ &= \mathrm{i}\,\frac{x_0}{x_1}\,\frac{1}{w(x_0/x_1)}\,(1+k) = \lambda_0. \end{aligned} \tag{4.44}$$

where we have used formula (4.45) and the relation $w(x_0/x_1) = 2\mathrm{i}az_0/(x_1x_2)$, which is a consequence of (4.4) and (4.6) for $t = 0$. (4.35). To end the calculation for the comparison of formulas (4.13) and (4.31) we need to derive relations between the complete elliptic integrals on Σ and Σ_1. Using *(ii)* and *(iv)* of Proposition 4.2 we have

$$\mathbf{K} = \int_0^1 \frac{dx}{w(x)} = \frac{1}{\mathrm{i}(1+k)}\int_0^\infty \frac{d\lambda}{\mu(\lambda)} = \frac{\mathbf{K}_1'}{\mathrm{i}(1+k)} \tag{4.45}$$

$$\mathbf{K}' = \int_0^\infty \frac{dx}{w(x)} = \frac{1}{\mathrm{i}(1+k)}\int_{\mathbf{o}_2}^{\mathbf{o}_1} \frac{d\lambda}{\mu(\lambda)} = \frac{2\mathbf{K}_1}{\mathrm{i}(1+k)}. \tag{4.46}$$

Substitution of (4.44), (4.45) and (4.46) in (4.13) now gives

$$-(1+k)x_2t = -i(1+k)u_0 + 2\mathbf{K}_1 + 4n\mathbf{K}_1 + 2\mathrm{i}mK_1'. \tag{4.47}$$

Using (4.35) in (4.31) and the equalities $k = x_1/x_2$, $\lambda_2 = (x_1+x_2)/2a$, we see that formulas (4.47) and (4.31) coincide.

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